**Module 3: Trigonometric Identities, Inverse Functions, and Applications**

**I. Trigonometric Identities**

After completing this section, you should be able to:

* state basic trigonometric identities including reciprocal, even-odd, cofunction, Pythagorean, sum, difference, and double-angle identities
* apply identities to manipulate expressions involving trigonometric functions and to find function values
* prove trigonometric identities

**A. Reciprocal, Ratio, Even-Odd, and Cofunction Identities**

An **identity** is an equation that is true for all possible inputs. (A **possible input** is a real number in the domain of the expressions on each side of the equation.)

For example, consider the equation *x*2 – 2*x* + 1 = (*x* – 1)2. The domain of the expression on each side of the equation consists of all real numbers. The equation is an identity because the equation is true for all real numbers *x*.

However, the equation *x*2 – 2*x* + 1 = *x*(*x* – 1) is *not* an identity. The equation is true for *x* = 1 but not for the domain of all real numbers.

In module 2, many relationships among trigonometric functions were encountered. In this module, these relationships will be stated as identities, and they will form the foundation for proving additional identities.

For example, for all *t* in the domain of the secant function; that is, the equation is true for all real numbers *t* except for multiples of π. The equation is one of the reciprocal relationships formulated in module 2, and it is an example of an identity. There are analogous reciprocal identities involving the cosecant and the cotangent. In addition, the tangent and cotangent are ratios of sine and cosine.

**Reciprocal Identities**



**Ratio Identities**



The graphs of trigonometric functions were studied in the previous module. Recall that a function *f* whose graph is symmetric with respect to the *y*-axis is called an *even function,* and satisfies*f*(–*t*) = *f*(*t*). A function *f* whose graph is symmetric with respect to the origin is called an *odd function,* and it satisfies *f*(–*t*) = –*f*(*t*). As discussed in module 2 topic IV, the cosine function is even, and the sine and tangent functions are odd. These properties lead to the following even-odd identities for sine, cosine, and tangent.

**Even-Odd Identities**

The cosine function is even: cos(–*t*) = cos *t*.  
The sine function is odd: sin(–*t*) = –sin *t*.  
The tangent function is odd: tan(–*t*) = –tan *t*.

Recall that the sine and cosine are examples of cofunctions. (See module 2, topic I.)

**Cofunction Identities**

|  |  |
| --- | --- |
| cos (π/2 – *t*) = sin *t* | sin (π/2 – *t*) = cos *t* |
| cot (π/2 – *t*) = tan *t* | tan (π/2 – *t*) = cot *t* |
| csc (π/2 – *t*) = sec *t* | sec (π/2 – *t*) = csc *t* |

These identities can be used in simplifying trigonometric expressions.

**Example I.A.1:**  Simplify the expression .

**Solution:**

|  |  |  |
| --- | --- | --- |
| First, simplify : | |  |
|  |  | Factor out –1. |
|  |  | Apply the identity sin(–*t*) = –sin *t* with . |
|  | = –cos *θ* | Apply the cofunction identity cos*t* with *t* *= θ.* |
|  |  |  |
|  |  | Factor out –1. Apply the ratio identity for the tangent, and the reciprocal identities for the cosecant and the secant. |
|  |  | Rewrite the division as multiplication. |
|  | = –cos *θ* | Simplify. |

**B. Pythagorean Identities and Tips for Proving Identities**

|  |  |
| --- | --- |
| Recall that any real number *t* corresponds to a point on the unit circle with coordinates (*x*, *y*) = (cos *t*, sin *t*).  The equation of the unit circle is *x*2 + *y*2 = 1.  For *x* = cos *t* and *y* = sin *t*, (cos *t*)2 + (sin *t*)2 = 1, or equivalently, (sin *t*)2 + (cos *t*)2 = 1. |  |

By convention, (sin *t*)2 is written sin2 *t* and (cos *t*)2 is written cos2 *t*, so the equation is stated assin2 *t* + cos2 *t* = 1.

**Caution:** Do not confuse with sin2 *t* with sin *t*2. sin2 *t* is the square of the sine of *t*, but sin *t*2 is the sine of the square of*t.*

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| The equation sin2 *t* + cos2 *t* = 1 is called a **Pythagorean identity**.  There is a connection with the Pythagorean theorem.  The reference triangle associated with the point (cos *t*, sin *t*) is a right triangle having legs of length |cos *t*| and |sin *t*| and hypotenuse of length 1.  Apply the Pythagorean theorem:   |  |  |  | | --- | --- | --- | | 1 | = | |sin *t*|2 + |cos *t*|2 | |  | = | sin2 *t* + cos2 *t* | |  |

The identity sin2 *t* + cos2 *t* = 1 can be used to establish additional Pythagorean identities involving the other trigonometric functions.

**Pythagorean Identities**

sin2 *t* + cos2 *t* = 1  
tan2 *t* + 1 = sec2 *t*cot2 *t* + 1 = csc2 *t*

**Example I.B.1:** Prove the Pythagorean identity tan2 *t* + 1 = sec2 *t*.

Proof:

|  |  |  |  |
| --- | --- | --- | --- |
| tan2 *t* + 1 |  |  | Start with the left side. Rewrite in terms of sine and cosine. |
|  |  |  | Write each term using the least common denominator, cos2 *t*. |
|  |  |  | Combine terms, since they have the same denominator. |
|  |  |  | Apply the identity sin2 *t* + cos2 *t* = 1. |
|  |  |  | Apply the power property of exponents. |
|  | = sec2 *t* |  | Apply the reciprocal identity . |

The equation tan2 *t* + 1 = sec2 *t* is true for any input *t* for which cos *t* is nonzero, which is precisely any input *t* in the domain of the tangent and secant functions.

The identity cot2 *t* + 1 = csc2 *t* can be proved in a similar fashion.

Example I.B.1 has illustrated one approach used to prove a trigonometric identity. This approach consisted of starting with the expression on one side of the equation, applying already-known identities and performing some algebraic manipulation, and then finally arriving at the expression on the other side of the equation.

Another approach is to treat each side of the equation separately, using already-known identities and performing algebraic manipulation until the end result for each side is identical.

**Tips for Proving a Trigonometric Identity**

1. Start with the more complicated side of the equation. Apply known identities and some algebraic manipulation. It is often helpful to convert an expression to sines and cosines. If you see how to carry out this process to arrive at the expression on the other side of the equation, you're done!
2. If the work on one side of the equation has gotten you to an expression *E*, but you don't easily see how to get from *E* to the expression on the other side of the equation, then stop your work at expression *E*. Now "attack" the expression on the other side of the equation, and see if you can apply identities and algebraic manipulation to arrive at expression *E*. If you can do this, then you have shown that both sides of the equation are equal to the same expression *E*, and you're done!
3. If neither step 1 nor step 2 has been fruitful, then you may need to apply some trial and error. Often, there are several identities that can be applied, and perhaps the identity you initially selected didn't pay off. Or, maybe you need to apply some more algebraic manipulation. For example, if you have a quotient with denominator of the form *a* – *b*, it may be helpful to multiply and divide by the quantity *a* + *b*.

The identities you will be asked to prove are amenable to these approaches.

**Example I.B.2:** Prove the identity .

**Proof:**

Since the right side is in terms of sine, start with the left side and write in terms of sines and cosines.

|  |  |  |
| --- | --- | --- |
| (sec *x* tan *x*)2 |  | Apply the reciprocal identity for secant and the ratio identity for tangent. |
|  |  | Combine terms. |
|  |  | Apply a property of exponents. |
| Now examine the right side of the equation: | | |
|  |  | Multiply and divide by (1 – sin *x*). |
|  |  | Simplify. |
|  |  | 1 – sin2 *x* = cos2 *x*, because 1 = sin2 *x* + cos2 *x.* |
| Both sides of the original equation have been shown to be equal to , so the identity has been proven. | | |

**C. Sum and Difference Identities**

There are many useful trigonometric identities. Indeed, soon you will have compiled a long list! There are important identities involving sums and differences.

**Sum and Difference Identities**

|  |  |
| --- | --- |
| sin(*u* + *v*) | = sin *u* cos *v* + cos *u* sin *v* |
| sin(*u* – *v*) | = sin *u* cos *v* – cos *u* sin *v* |
| cos(*u* + *v*) | = cos *u* cos *v* – sin *u* sin *v* |
| cos(*u* – *v*) | = cos *u* cos *v* + sin *u* sin *v* |

  


These identities can be helpful in determining exact trigonometric function values without the use of a calculator.

**Example I.C.1:** Find the exact value of .

**Solution:**

First, write  as a sum of two angles whose sine and cosine are known exactly:

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |  | Write the expression as a cosine of a sum. |
|  |  | Apply the sum identity for cosine. |
|  |  | Evaluate the cosines and sines. |
|  |  | Factor out . |
| **Alternative Method:** | | |
| First, write  as a difference of two angles whose sine and cosine are known exactly: | | |
|  |  | Write the expression as a cosine of a difference. |
|  |  | Apply the sum identity for cosine. |
|  |  | Evaluate the cosines and sines. |
|  |  | Factor out . |

Now consider the proofs of these sum and difference identities, starting with the identity associated with cos(*u* – *v*). The proof depends upon geometry and the distance formula. However, once this identity is established, the rest of the sum and difference identities are much simpler to prove.

**Example I.C.2:** Prove the difference identity for cosine: cos(*u* – *v*) = cos *u* cos *v* + sin *u* sin *v*.

**Proof:**

|  |  |
| --- | --- |
| Suppose that *u* and *v* are any real numbers. Recall that the real number *u* corresponds to an arc *u* on the unit circle and a rotation by an angle of *u* radians. An analogous statement applies to *v*.  On the unit circle, plot the corresponding points *A*: (cos *u*, sin *u*) and *B*: (cos *v*, sin *v*). |  |
| The points *A*, *B*, and origin *O* determine a triangle *AOB*. |  |
| Rotate the triangle counterclockwise *AOB* by –*v* radians to obtain a congruent triangle *COD*. (In the illustration, *v* is a positive number and so a rotation by –*v* is a rotation by *v*radians in the opposite direction: in this case, clockwise. | |
|  | |
| Because point *B* corresponds to a rotation of *v* radians, point *D* corresponds to a rotation of *v* – *v* = 0 radians, and so point *D* has coordinates (1, 0).  Because point *A* corresponds to a rotation of *u* radians, point *C* corresponds to a rotation of *u* – *v* radians, and so point*C* has coordinates (cos (*u* – *v*), sin (*u* – *v*)). |  |
| Because the triangles are congruent, the length of side *AB* and the length of side *CD* must be the same. Denote this length by *d*. Now apply the distance formula to calculate *d*. | |

For side *AB* with endpoints *A*: (cos *u*, sin *u*) and *B*: (cos *v*, sin *v*), the length ***d*** is given by .

|  |  |  |  |
| --- | --- | --- | --- |
| Therefore, | | | |
| *d*2 | = | (cos *u* – cos *v*)2 + (sin *u* – sin *v*)2 |  |
|  | = | (cos2 *u* – 2 cos *u* cos *v* + cos2 *v*) +     (sin2 *u* – 2 sin *u* sin *v* + sin2 *v*) | Square binomials. |
|  | = | (cos2 *u* + sin2 *u*) + (cos2 *v* + sin2 *v*) –     2(cos *u* cos *v* + sin *u* sin *v*) | Regroup. |
|  | = | 1 + 1 – 2(cos *u* cos *v* + sin *u* sin *v*) | Apply Pythagorean identity cos2 *t* + sin2 *t* = 1. |
|  | = | 2 – 2(cos *u* cos *v* + sin *u* sin *v*) | Add 1 + 1. |
| For side *CD* with endpoints *C*: (cos (*u* – *v*), sin (*u* – *v*)) and *D*: (1, 0), the length *d* is given by . | | | |
| Therefore, | | | |
| *d*2 | = | [cos (*u* – *v*) – 1]2 + sin2 (*u* – *v*) |  |
|  | = | cos2 (*u* – *v*) – 2 cos (*u* – *v*) +     1 + sin2 (*u* – *v*) | Square binomials. |
|  | = | [cos2 (*u* – *v*) + sin2 (*u* – *v*)] +     1 – 2 cos (*u* – *v*) | Regroup. |
|  | = | 2 – 2 cos (*u* – *v*) | Apply Pythagorean identity cos2 *t* + sin2 *t* = 1     for *t* = *u* – *v*. |
| Since the squared lengths of sides *AB* and *CD* are both equal to *d*2, then | | | |

|  |  |  |  |
| --- | --- | --- | --- |
| 2 – 2(cos *u* cos *v* + sin *u* sin *v*) | = | 2 – 2 cos (*u* – *v*) |  |
| –2(cos *u* cos *v* + sin *u* sin *v*) | = | –2 cos (*u* – *v*) | Subtract 2 from both sides. |
| cos *u* cos *v* + sin *u* sin *v* | = | cos (*u* – *v*) | Divide both sides by –2. |

The last equation is precisely the difference identity for the cosine, so you have successfully proved the identity.

Fortunately, the other sum and difference identities follow easily from the cosine difference identity.

**Example I.C.3:** Prove the sum identity sin (*u* + *v*) = sin *u* cos *v* + cos *u* sin *v*.

**Proof:**

|  |  |  |
| --- | --- | --- |
| sin (*u* + *v*) |  | Start with the left side. |
|  | = cos (π/2 – (*u* + *v*)) | Apply the cofunction identity     sin *t* = cos(π/2 – *t*), with *t* = *u* + *v*. |
|  | = cos ((π/2 – *u*) – *v*) | Regroup. |
|  | = cos (π/2 – *u*) cos *v* +         sin (π/2 – *u*) sin *v* | Apply the sum identity for cosine. |
|  | = sin *u* cos *v* + cos *u* sin *v* | Apply cofunction identities for cosine and sine. |

The final expression is the right side of the equation, so the identity is proved.

The sine identity for a difference and the tangent identities for a sum or difference can be proven in a similar fashion.

**Example I.C.4:** Prove that sin (π – *t*) = sin *t*.

**Proof:**

|  |  |  |  |
| --- | --- | --- | --- |
| sin (π – *t*) | = | sin π cos *t* – cos π sin *t* | Apply the difference identity for the sine. |
|  | = | 0 cos *t* – (–1) sin *t* | Evaluate sin π and cos π. |
|  | = | sin *t* | Simplify. |

Trigonometric identities prove helpful in many applications. For example, identities can be used to find trigonometric function values for angles that are sums or differences of special angles such as 30° and 45°.

**Example I.C.5:** Find the exact value of tan 75°.

Solution:

|  |  |  |  |
| --- | --- | --- | --- |
| tan 75° | = | tan (30° + 45°) | Write 75° as the sum of two known values, 30° and 45°. |
|  | = |  | Apply the sum identity for the tangent. |
|  | = |  | Evaluate the tangents. |
|  | = |  | Multiply the numerator and denominator by 3. |

The application of a trigonometric identity is helpful when graphing equations of the form *y* = *a* sin *t* + *b* cos *t*. By applying the sum identity for the sine, it is possible to algebraically [rewrite the expression *y* = *a* sin *t* + *b* cos *t* in the form *y* = *k* sin(*t* + *θ*)](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M3-Module_3/popups/popup3-1.html). The graph of *y* = *k* sin(*t* + *θ*) is a transformation of the sine function. (See module 2, topic V.)

**Graphing the Equation *y* = *a* sin *t* + *b* cos *t***

|  |  |  |
| --- | --- | --- |
| Write *a* sin *t* + *b* cos *t* | = | *k* sin(*t* + *θ*) |
| where *k* | = | and |

*θ* is the number (or angle) for which  and .

The graph of *y* = *k* sin(*t* + *θ*) is a transformation of the graph of *y* = sin *t*, having amplitude *k*, period 2π, and phase shift of –*θ*. (See module 2, topic V.)

**Example I.C.6:** Graph the equation *y* =cos *t* + sin *t*.

Solution:

Given *y* =cos *t* + sin *t*= *a* sin *t* + *b* cos *t*, *a* = and *b* = 1.

Calculate *k*:

.

Find *θ*:

 and 

 is the quadrant I angle having cosine equal to  and sine equal to .

Rewrite the expressioncos *t* + sin *t*:

|  |  |
| --- | --- |
| *y* | = cos *t* + sin *t* |
|  | *= k* sin(*t* + *θ*) with *k* = 2 and *θ* = |
|  | = |
|  | |
| The graph of *y* =  has amplitude 2, phase shift –, and period 2π. | |

|  |  |
| --- | --- |
| Graph *y* = :  Shift the graph of *y* = sin *t* to the left by  to arrive at the graph of *y* = . |  |
| Then, stretch the graph of *y* =  vertically by a factor of 2 to arrive at the graph of   |  |  |  | | --- | --- | --- | | *y* | = | cos *t* + sin *t* | |  | = |  | |  |

**D. Double-Angle and Half-Angle Identities**

Certain special cases of the sum and difference identities lead to double-angle identities and half-angle identities.

|  |  |
| --- | --- |
| **Double-Angle Identities** | **Half-Angle Identities** |
| |  |  |  | | --- | --- | --- | | sin 2*t* | = | 2 sin *t* cos *t* | | cos 2*t* | = = = | cos2 *t* – sin2 *t* 1 – 2 sin2 *t* 2 cos2 *t* – 1 | |  | = |  | |  |

**Example I.D.1:** Prove the double-angle identity sin 2*t* = 2 sin *t* cos *t*.

**Proof:**

|  |  |  |  |
| --- | --- | --- | --- |
| sin 2*t* | = | sin(*t* + *t*) | Write 2*t* as the sum of *t* and *t*. |
|  | = | sin *t* cos *t* + cos *t* sin *t* | Apply the sine sum identity with *u* = *t* and *v* = *t.* |
|  | = | 2 sin *t* cos *t* | Add the terms. |

**Example I.D.2:** Prove the double-angle identities for cosine.

|  |  |  |  |
| --- | --- | --- | --- |
| cos 2*t* | = | cos(*t* + *t*) | Write 2*t* as the sum of *t* and *t*. |
|  | = | cos *t* cos *t* – sin *t* sin *t* | Apply the sine sum identity with *u* = *t* and *v* = *t*. |
|  | = | cos2 *t* – sin2 *t* | The first formulation of the cosine identity is proved. |
|  | = | (1 – sin2 t) – sin2 *t* | Apply the identity cos2 *t* = 1 – sin2 *t*. |
|  | = | 1 – 2 sin2 *t* | Simplify. The second formulation of the cosine identity is proved. |
| Similarly, | | | |
| cos2 *t* – sin2 *t* |  |  |  |
|  | = | cos2 *t* – (1 – cos2 *t*) | Apply the identity cos2 *t* = 1 – sin2 *t*. |
|  | = | 2 cos2 *t* – 1 | Simplify. The third formulation of the cosine identity is proved. |

The half-angle identities follow directly from the double-angle identities for the cosine.

Given the identity cos 2*u* = 1 – 2 sin2 *u*, solve for sin2 *u* to obtain .

Apply the principle of square roots (see module 1, topic IV-A) to get .

Now set  to arrive at , which is one of the half-angle identities.

To arrive at the half-angle identity for the cosine, start with the identity cos 2*u* = 2 cos2 *u* – 1 and pursue a similar approach.

The double-angle and half-angle identities for the tangent can be established in a similar fashion.

**Example I.D.3:** Find the exact value of .

Solution:

|  |  |  |
| --- | --- | --- |
|  |  | Write . |
|  |  | Apply a half-angle identity for the tangent. |
|  |  | Evaluate cosine and sine. |
|  |  | Multiply numerator and denominator by 2. |

**Example I.D.4:** Find the exact value of sin 2*α* if tan *α* = –1/4 and *α*is a quadrant II angle.

Solution:

|  |  |
| --- | --- |
| Draw a reference triangle for angle *α* in quadrant II, with tan *α* = –1/4.  The hypotenuse has length . |  |
| sin 2*α* = 2 sin *α* cos *α* | Apply the double-angle identity for sine. |
|  | Refer to the triangle to determine sin *α* and cos *α*. |
| Recall that cos *α* is negative in quadrant II. Therefore, . | |

**Example I.D.5:** Prove the identity sin 3*θ* = sin *θ* (4 cos2 *θ* – 1).

Solution:

|  |  |  |  |
| --- | --- | --- | --- |
| sin 3*θ* | = | sin (*θ* + 2*θ*) | Write as the sine of a sum. |
|  | = | sin *θ* cos 2*θ* + cos *θ* sin 2*θ* | Apply the sum identity for sine. |
|  | = | sin *θ* (2cos2 *θ* – 1) + cos *θ* (2 sin *θ* cos *θ*) | Apply double-angle identities. |
|  | = | 2 sin *θ* cos2*θ* – sin *θ* + 2 sin *θ* cos2 *θ* | Multiply out the expression. |
|  | = | 4 sin *θ* cos2*θ* – sin *θ* | Combine terms. |
|  | = | sin *θ* (4 cos2*θ* – 1) | Factor out sin *θ*. |

[*Return to top of page*](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/Trigonometric%20Identities.html#pagetop)

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